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Résumé

This article is devoted to prove a stability result for two independent coefficients for a 2×2 nonlinear parabolic system with only one observation. The main idea to obtain this result is to use a modified form of the Carleman estimate given in [1].
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1 Introduction

This paper is an improvement of the work [1] in the sense that we determine two independent coefficients with the observation of only one component in a nonlinear 2×2 parabolic system.
Several works concern linear and nonlinear parabolic equations but few concern system of nonlinear parabolic equations. We can cite [6] and the references therein. Such systems arise in biological or ecological system (see [8], [9]).
Let $\Omega \subset \mathbb{R}^n$ be a bounded domain of $\mathbb{R}^n$ with $n \leq 3$ and $\omega \subset \Omega$ a non empty subset. We denote by ν the outward unit normal to Ω on $\Gamma = \partial\Omega$ assumed to be of class $\mathcal{C}^1$. Let $T > 0$ and $t_0 \in (0,T)$. We shall use the following notations $Q_0 = \Omega \times (0,T)$, $Q = \Omega \times (t_0,T)$, $Q_\omega = \omega \times (t_0,T)$, $\Sigma = \Gamma \times (t_0,T)$ and

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$\Sigma_0 = \Gamma \times (0,T)$. We consider the following 2×2 reaction-diffusion system :

$$\begin{cases} \partial_t U = \Delta U + a_{11}(x)U + a_{12}(x)V + a_{13}(x)f(U,V) & \text{in} \quad Q_0, \\ \partial_t V = \Delta V + a_{21}(x)U + a_{22}(x)V & \text{in} \quad Q_0, \\ U(x,t) = k_1(x,t),\ V(x,t) = k_2(x,t) & \text{on} \quad \Sigma_0, \\ U(x,0) = U_0 \quad \text{and} \quad V(x,0) = V_0 & \text{in} \quad \Omega, \end{cases} \tag{1}$$

where, the function f is assumed to be Lipschitz with respect the two variables U and V.
Uniqueness and existence results for initial boundary value problem for such systems can be found in [7].
Throughout this paper, we consider the following set

$$\Lambda(R) = \{\Phi \in L^\infty(\Omega);\ \|\Phi\|_{L^\infty(\Omega)} \leqslant R\},$$

where R is a given positive constant. For $t_0 \in (0,T)$, we denote $T' = \frac{t_0+T}{2}$.
Let (U,V) (resp. $(\widetilde{U}, \widetilde{V})$) be solution of (1) associated to (a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, k_1, k_2, U_0, V_0) (resp. (a_{11}, a_{12}, $\widetilde{a}_{13}$, $\widetilde{a}_{21}$, a_{22}, k_1, k_2, U_0, V_0)) satisfying some regularity and positivity properties :

Assumption 1.1. *1. For $i = 1,2$, $j = 1,2,3$, a_{ij}, $\widetilde{a}_{13}$ and $\widetilde{a}_{21} \in \Lambda(R)$.*

2. There exist constants $r_1 > 0$ and $a_0 > 0$ such that

$$\widetilde{U}_0 \geq r_1,\ \widetilde{V}_0 \geq 0,\ a_{21} \geq a_0,\ \widetilde{a}_{21} \geq a_0,\ a_{11}r_1 + a_{12}\widetilde{V}_0 + \widetilde{a}_{13}\, f(r_1, \widetilde{V}_0) \geq 0,\ k_1 \geq r_1 \text{ and } k_2 \geq 0.$$

Such assumptions allow us to state that the function $\widetilde{U}$ satisfies $|\widetilde{U}(x,T')| \geq r_1 > 0$ in Ω (see [10]).

Assumption 1.2. *1. The function f checks a generalized Lipschitz property in the following sense : $\exists\ C > 0$, such that $|\partial_t f(U,V) - \partial_t f(\widetilde{U},\widetilde{V})| \leq C\left(|U-\widetilde{U}| + |V-\widetilde{V}| + |(U-\widetilde{U})_t| + |(V-\widetilde{V})_t|\right)$.*

2. $\exists\ r_2 > 0$ such that $f(\widetilde{U},\widetilde{V})(T',x) \geq r_2 > 0$ in Ω.

3. $\partial_t f(U,V) \in L^2((0,T);H^2(\Omega))$.

This set of functions is not empty and contains, in particular, a large class of semilinear terms associated with ecological or biological models (e.g. $f(U,V) = U^\alpha V^\beta$ with α and β non negative constants).
The main result is the following Theorem :

Theorem 1.3. *Let ω be a subdomain of an open set Ω of $\mathbb{R}^n$. We suppose that Assumptions 1.1 and 1.2 are checked and $(U,V)(\cdot,T') = (\widetilde{U},\widetilde{V})(\cdot,T')$. Furthermore, we assume that $\widetilde{U}_0$, $\widetilde{V}_0$ in $H^2(\Omega)$. Then there exists a constant $C = C(\Omega,\omega,a_0,t_0,T,r_1,r_2,R) > 0$ such that*

$$\|a_{21} - \widetilde{a}_{21}\|^2_{L^2(\Omega)} + \|a_{13} - \widetilde{a}_{13}\|^2_{L^2(\Omega)} \leq C\|\partial_t V - \partial_t \widetilde{V}\|^2_{L^2(Q_\omega)}.$$

In [1], for a linear reaction diffusion system, we prove a stability result for one coefficient with only one observation. The novelty in this paper is the identification of two coefficients with only one observation for a nonlinear system. The main tool is a Carleman estimate established in [1] which is adapted to recover two independent coefficients, one in each equation of (1).
The paper is organized as follows : In section 2, we give the modified Carleman estimate for a reaction-diffusion system with only one observation. Then using this modified Carleman estimate, we prove in section 3 a stability result for two coefficients with the observation of only one component.

2 Carleman estimate

At first, we recall the general form of the Carleman estimate associated to the operator $\partial_t q - \Delta q$ (see [2], [4], [5]). Let $\omega' \Subset \omega \Subset \Omega$ and let $\widetilde{\beta}$ be a $\mathcal{C}^2(\Omega)$ function such that

$$\widetilde{\beta} > 0, \text{ in } \Omega,\ \widetilde{\beta} = 0 \text{ on } \partial\Omega,\ \min\{|\nabla\widetilde{\beta}(x)|,\ x \in \overline{\Omega \setminus \omega'}\} > 0 \quad \text{and} \quad \partial_\nu \widetilde{\beta} < 0 \text{ on } \partial\Omega.$$

Then, we define $\beta = \widetilde{\beta} + K$ with $K = m\|\widetilde{\beta}\|_\infty$ and $m > 1$. For $\lambda > 0$ and $t \in (t_0, T)$, we define the following weight functions (see [3])

$$\varphi(x,t) = \frac{e^{\lambda\beta(x)}}{(t-t_0)(T-t)}, \qquad \eta(x,t) = \frac{e^{2\lambda K} - e^{\lambda\beta(x)}}{(t-t_0)(T-t)}.$$

We have then the following Carleman estimate :

Theorem 2.1. *Let $\tau \in \mathbb{R}$. Then there exist $\lambda_0 = \lambda_0(\Omega,\omega) \geq 0$, $s_0 = s_0(\lambda_0, T, \tau) > 0$ and a positive constant $C_0 = C_0(\Omega, \omega, \tau)$ such that, for any $\lambda \geq \lambda_0$ and any $s \geq s_0$, the following estimate holds :*

$$I(\tau, q) \leq C_0 \left[\iint_{Q_\omega} e^{-2s\eta} \lambda^4 (s\varphi)^{\tau+3} |q|^2 \, dx \, dt + \iint_Q e^{-2s\eta} (s\varphi)^\tau |\partial_t q - \Delta q|^2 \, dx \, dt \right], \quad (2)$$

where

$$I(\tau,q) = \iint_Q e^{-2s\eta}(s\varphi)^{\tau-1}(|\partial_t q|^2 + |\Delta q|^2)\, dxdt + \lambda^2 \iint_Q e^{-2s\eta}(s\varphi)^{\tau+1}|\nabla q|^2 \, dxdt$$
$$+ \lambda^4 \iint_Q e^{-2s\eta}(s\varphi)^{\tau+3}|q|^2 \, dxdt$$

Remark 1. *If we denote*

$$M_1^{(\tau)}\psi = -\Delta\psi - s^2\lambda^2\varphi^2|\nabla\beta|^2\psi - (\frac{\tau}{2} - s\partial_t\eta)\psi, \quad M_2^{(\tau)}\psi = \partial_t\psi + 2s\lambda(\varphi + \frac{\tau}{2})\nabla\beta.\nabla\psi,$$

with $\psi = e^{-s\eta}\varphi^{\frac{\tau}{2}} q$, the Carleman estimate (2) also gives an estimation of $\|M_1^{(\tau)}\psi)\|^2_{L^2(Q)} + \|M_2^{(\tau)}\psi\|^2_{L^2(Q)}$.

We assume that $a_{11}, a_{12}, a_{21}, a_{22} \in \Lambda(R)$, $a_{21} \geq a_0 > 0$ and we consider the following system :

$$\begin{cases} \partial_t Y = \Delta Y + a_{11}(x)Y + a_{12}(x)Z + H_1, & \text{in} \quad Q_0, \\ \partial_t Z = \Delta Z + a_{21}(x)Y + a_{22}(x)Z + H_2 & \text{in} \quad Q_0, \\ Y(x,t) = Z(x,t) = 0 & \text{on} \quad \Sigma_0, \\ Y(x,0) = K_1, \quad Z(x,0) = K_2 & \text{in} \quad \Omega, \end{cases} \tag{3}$$

Then we can have, through the result given in [1], the following modified Carleman estimate with a single observation acting on a subdomain ω of Ω for the system (3) :

Theorem 2.2. *There exist $\lambda_1 = \lambda_1(\Omega, \omega) \geq 1$, $s_1 = s_1(\lambda_0, T) > 1$ and a positive constant $C_1 = C_1(\Omega, \omega, R, a_0)$ such that, for any $\lambda \geq \lambda_1$ and any $s \geq s_1$, the following estimate holds :*

$$\lambda^{-4} I(-3, Y) + I(0, Z) \leq C_1 s^4 \lambda^4 \iint_{Q_\omega} e^{-2s\eta} \varphi^4 |Z|^2 \, dx \, dt \tag{4}$$
$$+C_1 \left[s^{-3} \lambda^{-4} \iint_Q e^{-2s\eta} \varphi^{-3} \, |H_1|^2 \, dx \, dt + \iint_Q e^{-2s\eta} \, |H_2|^2 \, dx \, dt \right].$$

3 Stability result

Let (U, V) (resp. $(\widetilde{U}, \widetilde{V})$) be solution of (1) associated to (a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, k_1, k_2, U_0, V_0) (resp. (a_{11}, a_{12}, $\widetilde{a}_{13}$, $\widetilde{a}_{21}$, a_{22}, k_1, k_2, U_0, V_0). Then, if we set $u = U - \widetilde{U}$, $v = V - \widetilde{V}$, $Y = \partial_t u$ and $Z = \partial_t v$, (Y, Z) is solution to the following problem

$$\begin{cases} \partial_t Y = \Delta Y + a_{11}(x)Y + a_{12}(x)Z + \gamma_1 \partial_t f(\widetilde{U}, \widetilde{V}) + a_{13}(x) \partial_t F(U, V, \widetilde{U}, \widetilde{V}), & \text{in} \quad Q_0, \\ \partial_t Z = \Delta Z + a_{21}(x)Y + a_{22}(x)Z + \gamma_2 \partial_t \widetilde{U} & \text{in} \quad Q_0, \\ Y(x,t) = Z(x,t) = 0 & \text{on} \quad \Sigma_0, \\ Y(x,0) = \gamma_1 f(U_0, V_0), \quad Z(x,0) = \gamma_2 U_0 & \text{in} \quad \Omega, \end{cases} \tag{5}$$

where $\gamma_1 = (a_{13} - \widetilde{a}_{13})$, $\gamma_2 = (a_{21} - \widetilde{a}_{21})$ and $F(U, V, \widetilde{U}, \widetilde{V}) = f(U, V) - f(\widetilde{U}, \widetilde{V})$.

If we apply the modified Carleman estimate (4) to the previous system (5), we have

$$\lambda^{-4} I(-3, Y) + I(0, Z) \leq C_1 s^4 \lambda^4 \iint_{Q_\omega} e^{-2s\eta} \varphi^4 |Z|^2 \, dx \, dt$$
$$+C_1 \left[s^{-3} \lambda^{-4} \iint_Q e^{-2s\eta} \, \varphi^{-3} \, (|\gamma_1 \partial_t f(\widetilde{U}, \widetilde{V})|^2 + |\partial_t F|^2) \, dx \, dt + \iint_Q e^{-2s\eta} \, |\gamma_2 \partial_t \widetilde{U}|^2 \, dx \, dt \right]. \tag{6}$$

Now we shall "absorb" the term $A = s^{-3} \iint_Q e^{-2s\eta} \, \varphi^{-3} \, |\partial_t F|^2 \, dx \, dt$. So, we need the following lemma (see [6]) :

Lemma 3.1. *There exists a positive constant $C > 0$ such that.*

$$\iint_Q \left|\int_{T'}^t q(x,\xi)d\xi\right|^2 e^{-2s\eta}dxdt \leq \frac{C}{s}\iint_Q |q(x,t)|^2 e^{-2s\eta}dxdt$$

for all large $s > 0$ and $q \in L^2(Q)$.

Since $\varphi^{-3} \leq C\frac{T^6}{4^3}$, $\varphi^{-3} \leq C\frac{T^{12}}{4^6}\varphi^3$ and using Assumption 1.2-(1), the previous Lemma yields

$$A \leq Cs^{-3}(1+s^{-1})\iint_Q e^{-2s\eta}\ (|Y|^2 + \varphi^3|Z|^2)\ dx\ dt. \tag{7}$$

Therefore, for s and λ large enough, the integral A can be "absorbed" into the left hand side of (6).
Then (6) can be written as follows

$$\lambda^{-4}I(-3,Y) + I(0,Z) \leq C_1 s^4\lambda^4 \iint_{Q_\omega} e^{-2s\eta}\varphi^4|Z|^2\ dx\ dt \tag{8}$$

$$+C_1\left[s^{-3}\lambda^{-4}\iint_Q e^{-2s\eta}\ \varphi^{-3}\ |\gamma_1\partial_t f(\widetilde{U},\widetilde{V})|^2\ dx\ dt + \iint_Q e^{-2s\eta}\ |\gamma_2\partial_t\widetilde{U}|^2\ dx\ dt\right].$$

Let us introduce the following integral $\mathcal{I}_1 = \lambda^{-4}\int_{t_0}^{T'}\int_\Omega M_2^{(-3)}\psi_1\cdot\psi_1\ dx\ dt$, where $\psi_1 = e^{-s\eta}Y\varphi^{-3/2}$. We first estimate $\mathcal{I}_1$ with the modified Carleman estimate (8) :

$$\mathcal{I}_1 \leq \frac{1}{2}\lambda^{-2}\left[\lambda^{-4}\|M_2^{(-3)}\psi_1\|^2_{L^2(Q)} + \int_{t_0}^{T'}\int_\Omega e^{-2s\eta}\varphi^{-3}|Y|^2\ dx\ dt\right] \leq \frac{1}{2}\lambda^{-2}C\left[s^4\lambda^4\iint_{Q_\omega} e^{-2s\eta}\varphi^4|Z|^2\ dx\ dt\right.$$

$$\left. + s^{-3}\lambda^{-4}\iint_Q e^{-2s\eta}\ \varphi^{-3}\ |\gamma_1\partial_t f(\widetilde{U},\widetilde{V})|^2\ dx\ dt + \iint_Q e^{-2s\eta}\ |\gamma_2\partial_t\widetilde{U}|^2\ dx\ dt\right].$$

By computing $\mathcal{I}_1$, we obtain

$$\frac{1}{2}\lambda^{-4}\int_\Omega |\psi_1(x,T')|^2\ dx \leq 2|\mathcal{I}_1| + Cs\lambda^{-2}\iint_Q e^{-2s\eta}\ \varphi^{-2}|Y|^2\ dx\ dt$$

$$\leq C\lambda^{-2}\left[s^5\lambda^4\iint_{Q_\omega} e^{-2s\eta}\varphi^4|Z|^2\ dxdt + s^{-2}\lambda^{-4}\iint_Q e^{-2s\eta}\ \varphi^{-3}\ |\gamma_1\partial_t f(\widetilde{U},\widetilde{V})|^2\ dxdt\right.$$

$$\left. +s\iint_Q e^{-2s\eta}\ |\gamma_2\partial_t\widetilde{U}|^2\ dxdt\right]$$

Then, for λ sufficiently large and $\varphi^{-2} \leq CT^4$, we have

$$\lambda^{-4}\int_\Omega e^{-2s\eta(x,T')}\varphi^{-3}(x,T')|Y(x,T')|^2\ dx \leq C\left[s^5\lambda^2\iint_{Q_\omega} e^{-2s\eta}\varphi^4|Z|^2\ dx\ dt\right.$$

$$+ s^{-2}\lambda^{-6} \iint_Q e^{-2s\eta}\ \varphi^{-3}\ |\gamma_1 \partial_t f(\widetilde{U}, \widetilde{V})|^2\ dx\ dt + s\lambda^{-2} \iint_Q e^{-2s\eta}\ |\gamma_2 \partial_t \widetilde{U}|^2\ dx\ dt \Bigg]. \tag{9}$$

Moreover, since $(U, V)(\cdot, T') = (\widetilde{U}, \widetilde{V})(\cdot, T')$, we have $Y(x, T') = \gamma_1 f(\widetilde{U}, \widetilde{V})(x, T')$. Thus by [7], we can have $\widetilde{U} \in H^1((t_0, T); H^2(\Omega))$, so $\partial_t \widetilde{U} \in L^2((t_0, T); H^2(\Omega))$. Moreover, by Assumption 1.2-(3) $\partial_t f(\widetilde{U}, \widetilde{V}) \in L^2((t_0, T); H^2(\Omega))$. Then for $n \leq 3$, $\partial_t \widetilde{U}$ and $\partial_t f(\widetilde{U}, \widetilde{V})$ are in $L^2((t_0, T); L^\infty(\Omega))$ by classical Sobolev imbedding. Thus, using Assumption 1.2-(2) and $|\widetilde{U}(x, T')| \geq r_1 > 0$ in Ω, we obtain for λ sufficiently large :

$$\lambda^{-4} \int_\Omega e^{-2s\eta(x,T')} \varphi^{-3}(x, T') |\gamma_1|^2\ dx \leq C \left[s^5 \lambda^2 \iint_{Q_\omega} e^{-2s\eta} \varphi^4 |Z|^2\ dx\ dt + s\lambda^{-2} \iint_Q e^{-2s\eta}\ |\gamma_2|^2\ dx\ dt \right]. \tag{10}$$

In a similar way, we introduce $\mathcal{I}_2 = \int_{t_0}^{T'} \int_\Omega M_2^{(0)} \psi_2 \cdot \psi_2\ dx\ dt$, where $\psi_2 = e^{-s\eta} Z$. Thus, using the fact that $Z(x, T') = \gamma_2 \widetilde{U}(x, T')$, we obtain for λ sufficiently large

$$\int_\Omega e^{-2s\eta(x,T')} |\gamma_2|^2\ dx \leq C \left[s^{5/2} \lambda^2 \iint_{Q_\omega} e^{-2s\eta} \varphi^4 |Z|^2\ dx\ dt + s^{-9/2} \lambda^{-6} \iint_Q e^{-2s\eta}\ |\gamma_1|^2\ dx\ dt \right]. \tag{11}$$

So, if we add (10) and (11), we have

$$\lambda^{-4} \int_\Omega e^{-2s\eta} \varphi^{-3} |\gamma_1|^2\ dx + \int_\Omega e^{-2s\eta} |\gamma_2|^2\ dx \leq C s^5 \lambda^2 \iint_{Q_\omega} e^{-2s\eta} \varphi^4 |Z|^2\ dx\ dt.$$

Then, the proof of Theorem 1.3 is complete.

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